| Benha University<br>Faculty of Eng.- Shoubra<br>Eng. Math. & Phy. Department | | $1^{st}$ Year: Elec. Eng.(Power)<br>Mathematics 2-B<br>Date: 3 / 7 / 2011 |
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| Time 3 Hours | الامتحان (5) أسئلة فى صفحة واحدة و المطلوب إجابة كل الأسئلة | Marks |
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[1] Find the following integrals: — **20**

(a) $\int_0^\infty \frac{1}{\sqrt{xe^x}}\,dx$  (b) $\int_0^2 \frac{y^2}{\sqrt{2-y}}\,dy$  (c) $\int_0^{\pi/2} \sqrt{\cot z}\,dz$  (d) $\int_0^\infty \frac{2\sin 3t.\sin 4t}{t}\,dt$

[2](a) Find the series solution of the equation: $y`` - xy = 2x$ — **8**

(b) Using Laplace transforms, solve the equation: — **8**

$y`` - 3y` + 2y = e^{2t}, \quad y(0) = y`(0) = 0$

[3](a)Find the Laplace transformation of the functions: — **10**

(i) $f(t) = (e^{-t} - 2t)^2$  (ii) $f(t) = \sqrt{t} + e^{3t}\sin t$

(b)Find the inverse Laplace transform of : — **10**

(i) $F(s) = \frac{1}{s^2(s-1)}$  (ii) $F(s) = \frac{s}{s^2 - 3s + 2}$

[4] Solve the following partial differential equations: — **24**

(a) $u_x - 2u_y + 3u = 0, \quad u(0,y) = e^{3y}$  (b) $3u_x + 4u_y = 5(x^2 + y^2)$

(c) $u_{xx} - 3u_{xy} = e^{2x+y}$  (d) $u_{xx} - 3u_{xy} + 2u_{yy} = \cos(x+y)$

[5](a) Prove that: $B(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$ — **10**

(b)Solve the linear system: $\begin{bmatrix} x` \\ y` \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$ — **10**

*Good Luck*

*Dr. Mohamed Eid*

## Model Answer

[1] (a) $\int_0^\infty \frac{1}{\sqrt{xe^x}}\,dx = \int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{1}{2}x}\,dx$ . Put $x = 2y,\ dx = 2dy$

Then $I = \sqrt{2}\int_0^\infty y^{-\frac{1}{2}}\, e^{-y}\,dy = \sqrt{2}\Gamma(1/2) = \sqrt{2\pi}$

(b) $\int_0^2 \frac{y^2}{\sqrt{2-y}}dy$ . Put $y = 2x,\ dy = 2dx$

Then $I = \frac{8}{\sqrt{2}}\int_0^\infty x^2\,(1\text{-}x)^{-\frac{1}{2}}\,dx = 4\sqrt{2}\ B(3,\frac{1}{2}) = \frac{64\sqrt{2}}{15}$

(c) $\int_0^{\pi/2} \sqrt{\cot z}\ dz = \int_0^{\pi/2} (\cos z)^{\frac{1}{2}}.(\sin z)^{-\frac{1}{2}}\ dz = \frac{1}{2}B(\frac{1}{4},\frac{3}{4}) = \frac{\pi}{\sqrt{2}}$

(d) $\int_0^\infty \frac{2\sin 3t.\sin 4t}{t}\ dt$

Since $2\sin 3t\cos 4t = \cos t - \cos 7t$ and $L\{\cos t - \cos 7t\} = \frac{s}{s^2+1} - \frac{s}{s^2+49}$

Then $L\{\frac{\cos t - \cos 7t}{t}\} = \int_s^\infty (\frac{s}{s^2+1} - \frac{s}{s^2+49})ds = \frac{1}{2}\ln\frac{s^2+49}{s^2+1} = \int_0^\infty (\frac{\cos t - \cos 7t}{t})e^{-st}dt$

Putting $s = 0$, then $I = \frac{1}{2}\ln 49 = \ln 7$

[2](a) From the equation: $y`` - xy = 2x$

Since $p(x) = 0$ and $q(x) = -x$ are analytic functions at $x = 0$.

Then the power series solution takes the form: $y = \sum_{n=0}^{\infty} a_n x^n$

Then $y` = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y`` = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in the given equation, we get $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 2x$

Then $2 a_2 x^0 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 2x$

In the first sum, put n – 2 = m

In the second sum, put n + 1 = m

Then $2 a_2 + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} - a_{m-1}\right] x^m = 2x$

Equating the coefficients in both sides, we get

$2a_2 = 0$, then $a_2 = 0$

When $m = 1$: $6a_3 - a_0 = 2$ <u>Coefficient x</u>

$\left[(m+2)(m+1) a_{m+2} - a_{m-1}\right] = 0$, $m = 2, 3, 4, \ldots$

Thus the recurrence relation (R.R) is:

$$a_{m+2} = \frac{a_{m-1}}{(m+1)(m+2)}, \quad m = 2, 3 \ldots$$

If $m = 2$, then $a_4 = \frac{a_1}{12}$

If $m = 3$, then $a_5 = \frac{a_2}{20} = 0$

If $m = 4$, then $a_6 = \frac{a_3}{30} = \frac{2 + a_0}{180}$

Then $y(x) = a_0 + a_1 x + a_2 x^2 \ldots$

$$= a_0 + a_1 x + 0 + \frac{2+a_0}{6}x^3 + \frac{a_1}{12}x^4 + 0 + \frac{2+a_0}{180}x^6 + \ldots$$

$$= a_0[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \ldots] + a_1[x + \frac{1}{12}x^4 + \ldots] + [\frac{1}{3}x^3 + \frac{1}{90}x^6 + \ldots]$$

(b) Since $L\{ y`` - 3y` + 2y\} = L\{ e^{2t} \}$

Then $(s^2 Y - sy(0) - y`(0)) - 3(sY - y(0)) + 2Y = \frac{1}{s-2}$

From the conditions $y(0) = y`(0) = 0$

Then, we get $(s^2 - 3s + 2)Y = \frac{1}{s-2}$ Or $Y = \frac{1}{(s-1)(s-2)^2}$

Using methods of partial fractions, we get $Y = \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{(s-2)^2}$

Then, the solution of the equation $y(t) = e^t - e^{2t} + te^{2t}$

[3](a)(i) Since $f(t) = (e^{-t} - 2t)^2 = e^{-2t} + 4t^2 - 4te^{-t}$. Then $F(s) = \frac{1}{s+2} + \frac{8}{s^3} - \frac{4}{(s+1)^2}$

(ii)Since $f(t) = \sqrt{t} + e^{3t}\sin t$. Then $F(s) = \frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{(s-3)^2 + 1}$

(b)(i) $F(s) = \frac{1}{s^2(s-1)}$. Since $L^{-1}\{\frac{1}{s-1}\} = e^t$ and $L^{-1}\{\frac{1}{s(s-1)}\} = \int_0^t e^t dt = e^t - 1$

Then $f(t) = L^{-1}\{\frac{1}{s^2(s-1)}\} = \int_0^t (e^t - 1)dt = e^t - t$

(ii) Using methods of partial fractions, we get $F(s)=\dfrac{s}{s^2-3s+2}=\dfrac{2}{s-2}-\dfrac{1}{s-1}$

Then $f(t)=2e^{2t}-e^{t}$

[4](a) $u_x-2u_y+3u=0$, $u(0,y)=e^{3y}$

The required solution takes the form $u(x,y)=e^{ax+by}$. Then $u_x=au$, $u_y=bu$.

Substitute in the given equation, we get (a – 2b + 3)u = 0.

Then a – 2b + 3 = 0 and a = 2b – 3. Then $u(x,y)=e^{(2b-3)x+by}$

From the given condition, $u(0,y)=e^{3y}=e^{by}$. Then a = 3 = b.

Then the required solution is $u(x,y)=e^{3x+3y}$.

(b) $3u_x+4u_y=5(x^2+y^2)$

Let α = x cosθ + y sinθ and β = – x sinθ + y cosθ.

Then $u_x=u_\alpha\alpha_x+u_\beta\beta_x=u_\alpha\cos\theta-u_\beta\sin\theta$

$u_y=u_\alpha\alpha_y+u_\beta\beta_y=u_\alpha\sin\theta+u_\beta\cos\theta$

Substituting in the given equation, we get

$$3(\cos\theta.u_\alpha-\sin\theta.u_\beta)+4(\sin\theta.u_\alpha+\cos\theta.u_\beta)=5(\alpha^2+\beta^2)$$

Since u(x , y) = w(α , β). Then

$$\left[3\cos\theta+4\sin\theta\right]w_\alpha+\left[-3\sin\theta+4\cos\theta\right]w_\beta=5(\alpha^2+\beta^2)$$

If the coefficient of $w_\beta$ is zero, that is, – 3sinθ + 4cosθ = 0.

Then $\tan\theta = \frac{4}{3}$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

Then, we get $w_\alpha = \alpha^2 + \beta^2$

$$w = \int(\alpha^2 + \beta^2)d\alpha = \frac{1}{3}\alpha^3 + \alpha\beta^2 + c(\beta).$$

Then $u(x, y) = \frac{1}{3}(\frac{3x+4y}{5})^3 + \frac{3x+4y}{5}(\frac{-4x+3y}{5})^2 + c(\frac{-4x+3y}{5})$

where $c(\frac{-4x+3y}{5})$ is arbitrary function.

(c) $u_{xx} - 3u_{xy} = e^{2x+y}$. Since the C.E. is $k^2 - 3k = 0$. Then $k = 0$, $k = 3$.

Then $u_c = f_1(y + 0x) + f_2(y + 3x)$

$$u_I = \frac{1}{D^2 - 3DE}e^{2x+y} = \frac{1}{4-6}e^{2x+y} = \frac{1}{-2}e^{2x+y}$$

The general solution is $u(x, y) = u_c + u_I$

(d) $u_{xx} - 3u_{xy} + 2u_{yy} = \cos(x + y)$. Since the C.E. is $k^2 - 3k + 2 = 0$. Then $k = 1$, $k = 2$.

Then $u_c = f_1(y + x) + f_2(y + 2x)$

$$u_I = \frac{1}{D^2 - 3DE + 2E^2}\cos(x + y) = \frac{1}{-1+3-2}\cos(x + y) = \frac{1}{(D-E)(D-2E)}\cos(x + y)$$

Assume that $y + x = c_1$ and $y + 2x = c_2$. Then

$$\frac{1}{(D-E)}\cos(x + y) = \int\cos(x + c_1 - x)dx = \int\cos c_1 dx = x\cos c_1 = x\cos(x + y)$$

$$\frac{1}{D-2E}x\cos(x+y) = \int x\cos(x+c_2-2x)dx$$

$$= \int x\cos(c_2-x)dx \qquad \text{(Integrate by parts)}$$

$$= x\sin(c_2-x)+\cos(c_2-x)$$

$$= x\sin(y+x)+\cos(y+x)$$

Then $u_I = x\sin(y+x)+\cos(y+x)$

The general solution is $u(x, y) = u_c + u_I$

[5](a) Theorem: $B(m,n) = \dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

(b) The linear system: $\begin{bmatrix} x` \\ y` \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$

The coefficient matrix: A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Then |A – mI| = $\begin{vmatrix} 1-m & 2 \\ 2 & 1-m \end{vmatrix} = (1–m)^2 - 4 = m^2 - 2m - 3 = 0$. Then m = 3, – 1.

The characteristic value problem is:

$$\begin{bmatrix} 1-m & 2 \\ 2 & 1-m \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If m = 3, then we get the linear system: $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get: 2a – 2b = 0. Putting b = 1, we get a = 1.

Then, the eignvector is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t}$

If $m = -1$, then we get the linear system: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get: $2a + 2b = 0$. Putting $b = 1$, we get $a = -1$.

Then, the eignvector is $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$

The fundamental matrix is $X = \begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}$

Then $|X| = 2e^{2t}$

$$X^{-1} = \frac{1}{2e^{2t}}\begin{bmatrix} e^{-t} & e^{-t} \\ -e^{3t} & e^{3t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$X^{-1}f(t) = \frac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-2t} + e^{-4t} \\ -e^{2t} + 1 \end{bmatrix}$$

$$\int (X^{-1}f(t))dt = \frac{1}{2}\begin{bmatrix} \frac{-1}{2}e^{-2t} - \frac{1}{4}e^{-4t} \\ -\frac{1}{2}e^{2t} + t \end{bmatrix}$$

$$v(t) = X\int (X^{-1}f(t))dt = \frac{1}{2}\begin{bmatrix} e^{3t} & -e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\begin{bmatrix} \frac{-1}{2}e^{-2t} - \frac{1}{4}e^{-4t} \\ -\frac{1}{2}e^{2t} + t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (-t - \frac{1}{4})e^{-t} \\ (t - \frac{1}{4})e^{-t} - e^{t} \end{bmatrix}$$

The general solution is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1X_1 + c_2X_2 + v(t)$